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Research Article

ANALYTICAL PERSPECTIVES ON RIEMANN ZETA ZEROS THROUGH TOY MODEL FRAMEWORKS

Prof. Amina Folashade Adeyemi

Independent Researcher, Mathematical Sciences, Orogbum, Nigeria

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Abstract

This study revisits a toy model framework for exploring the non-trivial roots of the Riemann zeta function and clarifies aspects of earlier works in a continuing research series on the Riemann Hypothesis. Root model equations are constructed via Taylor series expansions to provide insight into the behavior of the zeros and facilitate analytical and computational verification. Numerical contour integral methods (e.g., MATLAB) and telescopic techniques are proposed to verify known zeros, while a mathematical-linguistic method organizes logical inferences into a structured framework called a "Logical and Organized Context." This combined approach offers a conceptual pathway for analyzing complex conjectures and may have broader implications in cryptography, aerodynamics, and computational linguistics.

Keywords: Riemann Zeta Function, Riemann Hypothesis, Non-trivial Zeros, Toy Model Framework, Mathematical-Linguistic Method

INTRODUCTION

With reference to my prior discussions in my series of papers for the Riemann Hypothesis [1], [2], [3], [4] & [5], there may be still some defects which need amendments, modifications and clarifications in the present paper. In reality, the first priority of these papers is to establish a root model equations from the well-known Riemann Zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$ by the Taylor series [1].

Certainly, we may select all of the non-trivial zeta roots and thus establish the corresponding root model equation for a further evaluation and comparison etc. This author will discuss such matter in my future papers of the Riemann research series. Next, this author will try to verify those known non-trivial zeta roots by some Mat lab contour integral coding as well as the telescopic method to prove the Riemann Hypothesis in a pioneer way. Lastly, this author will employ the mathematical linguistic method and turn some corners round to show that the Riemann Hypothesis is true. Or we may have solved the Riemann Hypothesis issue completely through a 4 cases of truth table collaborations with the logical inference [7] & [15] to the causality (structural) organization of the collected sentences [16] for the formulation of a particular meaningful context or just named as "Logical & Orgatizated Context". In practice, such kind of logical organization may be used in handling those conjectures or hypothesis like the continuum one or the Riemann one which has applications in aerodynamics or the cryptography or many other kinds of everyday life usage like the present hot topic of Natural Language Processing in the field of computer engineering.

A Supplementary Revise to this author's Previous Research
1. for the Root Model Equation of the Riemann Zeta Function [1]

$\frac{-3k+x}{2(-x+k)} - \frac{1}{2}\frac{x-3k}{(-x+k)}$

$= \frac{1}{2}\left[\frac{x-3k}{-(x-k)}\right]$

$= \frac{1}{2}\left|-\left[1-\frac{2k}{x-k}\right]\right|$ when we consider it as the locus of circle with centre (1,0) and

1

radius $\frac{}{2}$

$= \frac{1}{2}$ as x $\rightarrow \infty$

or

$\left|\frac{-3k+x}{2(-x+k)}\right| = \frac{3}{2}$ as k $\rightarrow \infty$ (rejected as $\zeta\left(\frac{3}{2}+vi\right) \neq 0$ for all residue values of v as verified by the Matlab code in [3] are zeros together with 1.5 lies on the zero free region [25]).

Also, geometrically, the Riemann Hypothesis may be solved [5] by the DeMoivre's Theorem of the God's Equation: $e^{\pi i} + 1 = 0$ with the paired complex roots of conjugate for the complex-valued roots of unity equation. That say, there may be some ξ_x or structures existing between two consecutive ξ_i and ξ_j such that $\oint B \cdot A \neq 0$. If the real part of such ξ_x is NOT 0.5, then our cognition about the non-trivial zeros of Riemann Hypothesis may need to be reconstructed. Although we may show that there may be another feasible root (u = 1.5) in the second order of the Taylor approximation to the Riemann Zeta function, contour integral (3/2 + vI) always equals to zero$\forall v \in \mathbb{R}$ [], $\zeta\left(\frac{3}{2}+vi\right) \neq 0$ without any structures lying on the line x = 3/2. In between $\frac{1}{2}$ and $\frac{3}{2}$, there is a singularity at u = 1 which is obviously a zero free zone or no zeros on the line x = 1. In fact, by considering the Dirichlet Eta function [12], [13] & [14] where $\eta(s) = (1-2^{1-s})\xi(s)$, then all of the non-trivial zeta zeros are also the roots of $\eta(s)$. Similarily, by constructing an artifical function, say $\eta_1(s) = (1.5-s)\xi(s)$, then all of the non-trivial zeta zeros are also the roots of $\eta_1(s)$ with the additional zeros at x = 1.5 that may be viewed as another singularity structure just like the case of x = 1. The focus of the present paper is NOT on the feasible singularity structures of $\eta_1(s) = (1.5\text{-}s)\,\xi(s)$.

In reality, $H_x = \sum_{k=1}^{x}\frac{1}{k}$= cot(x) and $\xi(\sigma) = \sum_{k=1}^{n}\frac{1}{k^{u\pm vi}}$ where σ = u+/-vi and $u, v \in \mathbb{R}$

$\sum_{k=1}^{n}\frac{1}{k^{u+vi}} = \sum_{k=1}^{n}\cdot\frac{1}{k^{u}}\frac{1}{k^{vi}}$

$= \sum_{k=1}^{n}\cdot\frac{1}{(u)e^{viln(k)}}$,

then $\Theta = \pm v ln(k)$.

Thus, the angle difference or the argument between H_x and $\xi(\sigma)$ is:

$\cot\left(x - arctan(\pm vln(k))\right) = \frac{cot(x)cot(arctan(\pm vln(k)))-1}{cot(x)+cot(arctan(\pm vln(k)))}$

$= \frac{\frac{cotx \pm vln(k)}{\pm vln(k)}}{\frac{\pm vln(k)cotx+1}{\pm vln(k)}}$

$= \frac{cotx \pm vln(k)}{\pm vln(k)cotx+1}$

In practice, there are infinite many root model equation for the above cotangent equation spreads over 360^o or 2nπ. To cite two cases: 0 & ∞,

For $\frac{cotx \pm vln(k)}{\pm vln(k)cotx+1}$= 0, cot x = $\pm v(k)$ or v = $\pm \frac{cotx}{ln(k)}$.

For $\frac{cotx \pm vln(k)}{\pm vln(k)cotx+1} = \infty$, $\pm vln(k)cotx$ = -1 or v = $\pm \frac{tanx}{ln(k)}$.

Practically, $\frac{d}{dx}\frac{ln(x)}{tan(ln(x))} = \frac{1}{xtan(ln(x))}$ - $\frac{ln(x)[1+tan^2(ln(x))]}{xtan^2(ln(x))}$

$= \frac{cot(ln(x))}{x}$ - …

$= \frac{ln(ln(x))}{x}$

(N.B. By solving $c(ln(x))$ = 0, one may get x = $e^{\pi/2}$ or x = $e^{-\pi/2}$ or approximate the roots of $\frac{ln(x)}{tan(ln(x))}$.)

$\frac{d}{dx} ln(ln(x)) = \frac{1}{xln(x)} . \approx \frac{1}{tanx(ln(x))}$ by small angle approximation $x \approx tanx$

$= \frac{cotx}{ln(x)}$

But when we take $ln(x) = \frac{2(x-1)}{(x+1)}$, then $ln(ln(x)) = \frac{2(x-3)}{(3x+1)}$,

Moreover, we may have three proposed Riemann Zeta Root Equation Model (RZREM):

$\frac{d}{dx}\left(-cot(ln(x))\right) = \frac{1}{tan(x)}\frac{1}{ln(x)} = \frac{cotx}{ln(x)}$ ----- (Case i: New Expected RZREM)

$\frac{d}{dx}\frac{4cot(ln(x))}{(x+1)^2} = \frac{-8*cot(ln(x))}{(x+1)^3} + \frac{4*(-1-cot(ln(x))^2)}{((x+1)^2*x)}$ ----- (Case ii: Old Proposal RZREM)

$\frac{d}{dx}\frac{ln(ln(x))}{x} = \frac{1}{(x^2 ln(x))} - \frac{ln(ln(x))}{x^2}$ ----- (Case iii: Modified Approximated RZREM)

(N.B. $\frac{4cot(ln(x))}{(x+1)^2}$ is a good approximation to $\frac{ln(ln(x))}{x^2}$ as $cot(ln(x)) \approx ln(ln(x))$ and $(x+1)^2 \approx x^2$.

The only thing that we may need is to fine adjust the empirical Riemann Zeta Root Equation $\frac{4cot(ln(x))}{(x+1)^2}$ as shown in the coming section. Practically, the model equation $\frac{4cot(ln(x))}{(x+1)^2}$ is NOT a wrong model as the radius of convergence is 0.79 which is smaller than 1. To be precise, it is just one of the feasible (empirical) model equation for Riemann Zeta non-trivial roots. We may thus employ Convex Optimization or Lagrange Multiplier method to fine adjust it. This author will discuss in my coming papers of the same Riemann research series.)

Case i: New Expected RZREM

When we employ the Taylor Series cot x = $_{x}$ and $ln(x) = \frac{2(x-1)1}{(x+1)} - \frac{x}{3}$ for

$$fsolve\left(\left|\left(\frac{(-x^2+3)}{(3*x)}\frac{(x+1)}{(2*(x-1))}\right)\right| - |(0.0004319*x^9 - 0.02216*x^8 + 0.4869*x^7 - 5.986*x^6 + 45.1*x^5 - 214.5*x^4 + 638.4*x^3 - 1134*x^2 + 1080*x - 395.2)|, x, -\infty..0\right)$$

The radius of convergence (value) is -0.001257674845 or | 0.001257674845|.

Case ii & iii (may act as a control to Case I):

$$fsolve\left(\left(\left(-8*\frac{cot(ln(x))}{(x+1)^3} + \frac{4*\left(-1-cot(ln(x))^2\right)}{((x+1)^2*x)}\right) + (-0.0004319*x^9 + 0.02216*x^8 - 0.4869*x^7 + 5.986*x^6 - 45.1*x^5 + 214.5*x^4 - 638.4*x^3 + 1134*x^2 - 1080*x + 395.2), x, 0..\infty\right)\right.$$

The answer for case ii is: 6.368727287

Also,

$$fsolve\left(\left(\left(\frac{1}{(x^2*ln(x))} - \frac{ln(ln(x))}{x^2}\right) + (-0.0004319*x^9 + 0.02216*x^8 - 0.4869*x^7 + 5.986*x^6 - 45.1*x^5 + 214.5*x^4 - 638.4*x^3 + 1134*x^2 - 1080*x + 395.2), x, 0..\infty\right)\right.$$

The answer for Case iii is: 6.368593478

Both of the solved answers are nearly the same if we rounded them to 3 decimal places or approximate by 6.369. This outcome implies that case ii & case iii indeed may basically be considered as the same Riemann Zeta Root Equation Model.

If we try to compute the radius of convergence (value), we may get:

$$fsolve\left(\left|\left(-8*\frac{cot(ln(x))}{(x+1)^3}+\frac{4*\left(-1-cot(ln(x))^2\right)}{((x+1)^2*x)}\right)\right| - |(0.0004319*x^9-0.02216*x^8+0.4869*x^7-5.986*x^6+45.1*x^5 - 214.5*x^4+638.4*x^3-1134*x^2+1080*x-395.2)|, x, 0..\infty\right)$$

The radius of convergence (value) for Case ii is: 0.01038875224

$$fsolve\left(\left|\left(\frac{1}{(x^2*ln(x))}-\frac{ln(ln(x))}{x^2}\right)\right| - |(0.0004319*x^9-0.02216*x^8+0.4869*x^7-5.986*x^6+45.1*x^5-214.5*x^4+638.4*x^3-1134*x^2+1080*x-395.2)|, x, -\infty..0\right)$$

The radius of convergence (value) for Case iii is: -0.07725137438 or | 0.07725137438|.

$cotx$

In brief, it seems that the best Riemann Zeta Root Model Equation is Case i or $\frac{}{ln(x)}$. The result is therefore consistent with what we may expect as the radius of convergence for it is the smallest among the three cases. In reality, all of the other feasible Riemann Zeta Root Model equation may follow the above procedure in order to verify its radius of convergence value and check for their feasibility. Thus, this author will not repeat for once more. In fact as the true expected model is different from the old proposed model, we therefore may need to fine adjusted the old proposed model $\frac{4cot(ln(x))}{(x+1)^2}$ so as achieve its best convergence value together with a suitable Lagrange Multiplier procedure for a best optimized value between the interpolated first true ten imaginary parts of the Riemann Zeta roots' associated polynomial etc.

Finally, $\frac{4cot(ln(x))}{(x+1)^2}$ is one of the feasible Riemann Zeta Root Equation Model (for the imaginary part)

$cotx$ but **NOT the best model NOR the wrong model** such as the one $\frac{}{ln(x)}$ with the least of radius of convergence value only. When we compare the above model $\frac{4cot(ln(x))}{(x+1)^2}$ with $\frac{-8*cot(ln(x))}{(x+1)^3}+$ $\frac{4*\left(-1-cot(ln(x))^2\right)}{((x+1)^2*x)}$, the difference in the first term is 4 and -8 with the $(x+1)^2$ and $(x+1)^3$. That is why my old proposed model may be a good approximation to the new expected model of the Riemann Zeta Root model. But we may still need to fine adjusted the old proposed model in the next section. Indeed, for

$\xi(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s}$ ---------------- (1)

by employing the Taylor Series from the Canada Maple Soft Maple, we may finally get the corresponding root model: s =

$$\frac{\left(\pi^2a^2 + 2I\pi au + a\pi I - 2\pi av - 2Iuv - 2vI - u^2 + v^2 + \sqrt{\pi^2a^2 + 2I\pi au - 2\pi av - 2Iuv - 2vI - u^2 + v^2}\right.}{\pi^2a^2 + 2I\pi au - 2\pi av - 2Iuv - vI - u^2 + v^2 - u}$$

(N.B. As $\eta(s) = \sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s} = (1 - 2^{1-s})\xi(s)$, then we may have:

$\frac{\eta(s)}{\xi(s)} = (1 - 2^{1-s})$ or $\frac{\eta(s)}{\xi(s)} + 2^{1-s} = 1$.

Hence, $(1.5 - s) = \frac{\eta(s)}{\xi(s)} + 2^{1-s} + \frac{1}{2} - s$ or

$3\text{-}2s = 2\frac{\eta(s)}{\xi(s)} + 2^{2-s} + 1 - 2s$,

i.e. In terms of η(s) and ξ(s), we may get:

$\eta_1(s) = (3\text{-}2s)\xi(s)$

$= 2\eta(s) + [2^{2-s} - 2s + 1]\xi(s)$

$= 2\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s} + [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s}$

$\eta_1(s) = [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s} + 2\sum_{n=1}^{\infty}\frac{1}{n^s}$(for n-1 is even, n is odd) or $\eta_1(s) = [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s} - 2\sum_{n=1}^{\infty}\frac{1}{n^s}$ (for n-1 is odd, n is even)

$\eta_1(s) = [2^{2-s} - 2s + 3]\sum_{n=1}^{\infty}\frac{1}{(2n+1)^s}$ or $\eta_1(s) = [2^{2-s} - 2s - 1]\sum_{n=1}^{\infty}\frac{1}{(2n)^s}$.

If we employ one of the new expected root model equation $\frac{cot(x)}{ln(x)}$ with the Taylor approximation

$\frac{(-x^2+3)}{(3*x)}\frac{(x+1)}{(2*(x-1))}$ when x = 2n is even, then we may have:

$\eta_1(s) = [2^{2-s} - 2s - 1]\frac{(3-4s^2)}{(3*2s)}\frac{(2s+1)}{(2*(2s-1))}$ where $\eta_1(s) = 0$ at $s = \frac{-1}{2}$ or $s = \pm\frac{\sqrt{3}}{2}$ or

$[2^{2-s} - 2s - 1] = 0$. Moreover, $[2^{2-s} - 2s - 1] = (2^{2-s} - s^2) = -5.07799867026$ when $s = 1 + \sqrt{2}I$.

Similarly, one may calculate $s = 1 - \sqrt{2}I$ for those who feel interested by themselves and this author will NOT repeat. In addition, the result is also a real-valued number. Also, the outcome is similar for a same process when x = 2n + 1 or odd number for

$[2^{2-s} - 2s + 3] = (2^{2-s} - s^2) = -2.11361628 - 4.48970939I$ when $s = 1 + \sqrt{2}I$. Similarly, one may

calculate $s = 1 - \sqrt{2}I$ for those who feel interested by themselves and this author will NOT repeat. In addition, the result is also a complex valued number. In practice, the above computational procedure is what the process for root of root at $\eta_1(s) = 0$. Obviously, the above computation is similar to what the symmetric properties of the Dirichlet Eta functional equation etc [12],[13] & [14]. In addition, as the aforementioned function $\eta_1(s)$ contains odd and even function, they can be expressed in terms of sine and cosine functions or in the form of the Quantum fourier transform [17]. Then we may calculate the corresponding quantum fourier series [18], [19], [20] & [21] which may be used to minimize a quantum system or computer's noise (but NOT applied in the social political areas) [2]. To go a forward step, one may use the HKLam statistical model theory to establish the linear regression models [23], [24] for some necessary quantum gate's matrix representation [22]. Hence, we may develop the respective quantum algorithm(s) for simulating some feasible quantum device etc. However, this may belong to the field of quantum computing or engineering, which is out of the focus of the present paper. (This author may leave the above discussion in my future papers when time & conditions are available.)

In reality, this author wants to remark that:

$$\eta_1(s) = 2\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s} + [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$= 2\,\frac{\sum_{n=1}^{\infty} e^{(I(n-1)\pi)}}{n^s} + [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$= -2 + [2^{2-s} - 2s + 1]\sum_{n=1}^{\infty}\frac{1}{n^s} = [2^{2-s} - 2s - 1]\sum_{n=1}^{\infty}\frac{1}{n^s}$$

But $[2^{2-s} - 2s - 1]$ can be solved by the Lambert function from the Maple Soft Personal Edition. This may in fact give:

$[2^{2-s} - 2s - 1]\xi(s)$ where $s = 0.7168497884$ together with the non-trivial roots of zeta.

Consequencely, for

$(0.7168497884 - s) = \frac{\eta(s)}{\xi(s)} + 2^{1-s} + 0.2831502116 - s$

By repeating the above process and the Maple Soft Personal Edition, we may get:

$[2^{1-s} - 0.2831502116s - 1]\xi(s)$ where $s = 0.8366974768$ together with the non-trivial roots of zeta.

Repeating the above process once more, we may get $s = 0.9048637552$, Repeat once more, $s = 0.9442575914$; After repeating the above process for five more time, $s = 0.9960360493$

Obviously, s will tend to 1 if we continue the above process for a few more times.

In fact, the sequence for $s = \{0.7168, 0.8367, 0.9049, 0.9442, \ldots, 0.9960, \ldots\}$ converges to 1. In other words, we may approximate any $\eta_x(s) = (x-s)\xi(s)$ by the above procedure obtain the respective converaging sequence. We may define the sequence $\{a_n\}_{1}^{\infty}$ by: $a_{n+1} = a_n + \frac{a_n}{6n}$

As $\frac{a_n}{6n} \to 0$ when $n \to \infty$, then $a_{n+1} \to a_n$ or $\lim_{n\to\infty} a_n = 1$.

Radius of Convergence = $\frac{a_1}{a_2} = \frac{0.7168497884}{0.8366974768} = 0.8567610257 < 1$

Obviously, given any real number with the decimal places, we may find its corresponding Taylor expansion for its product-log or by our Mathematica Home Edition, we may have:

In: Series[ProductLog[x], {x, 1.5, 1}]

Out: $ProductLog\left[\frac{3}{2}\right] + \frac{2ProductLog\left[\frac{3}{2}\right](x-1.5)}{3\left(1+ProductLog\left[\frac{3}{2}\right]\right)} + o(x-1.5)^2$

As $ProductLog\left|\frac{3}{2}\right| = 0.72586136$, we may have the following Taylor Expansion:

$$0.72586136 + \frac{2(0.72586136)(x-1.5)}{3(1+0.72586136)} + o(x-1.5)^2$$

We may also get a similar result for the values in both real parts and imaginary parts of any given complex numbers or even the non-trivial zeta zeros. In other words, we may approximate all of the known non-trivial zeta zeros by the Taylor Expansion of the Product-Log through Mathematica. By the way, given any Taylor series of the Product-Log, we may find out its real or complex value representation.

A last few words, when we are talking about the application, the above research Taylor series of the Product-Log may be used in the electric circuit analysis together with the commercial pursuit affairs etc. But they are out of the present paper's focus. This author will leave those discussions to other interested parties.

2. For the Case {s = u + vI where $u \in \mathbb{R}/\{0.5\}$} [2], [4] & [5]

As W = $\frac{(|u-v*cot(x+1)|)}{(r*|u*cot(x+1)+v|)}$,

$$\frac{\partial W}{\partial x} = \frac{([-v*(-1-cot(x+1)^2)])}{(r*[u*cot(x+1)+v])} - \frac{[u-v*cot(x+1)]*[0]+r*[u*(-1-cot(x+1)^2)]}{r*[u*cot(x+1)+v]^2}$$

$$= \frac{([-v*(-1)csc^2(x+1)])*(r*[u*cot(x+1)+v])-[u-v*cot(x+1)]*\left[[0]+r*[u*(-1-cot(x+1)^2)]\right]}{r*[u*cot(x+1)+v]^2}$$

$$= \frac{([-v*(-1)*csc^2(x+1)])*(r*[u*cot(x+1)+v])-[u-v*cot(x+1)]*r*[u*(-1)*csc^2(x+1)]}{r*[u*cot(x+1)+v]^2}$$

$$= csc^2(x+1)\frac{\left[[v(r(ucot(x+1)+v))]-[u-vcot(x+1)]*(-ru)\right]}{r*[u*cot(x+1)+v]^2}$$

$$= \frac{r(u^2+v^2)csc^2(x+1)}{r*[u*cot(x+1)+v]^2} = \frac{(r^2)csc^2(x+1)}{[u*cot(x+1)+v]^2}$$

$\frac{}{\partial x}\partial W$ $^2(x+1) = 0$ or $cot^2(x+1) = -1$

For = 0, we may have csc

i.e. $c(x+1) = \pm I$

$\frac{1}{x+1} - \frac{x+1}{3} = \pm I$ or

- x^2 + (-2 ±3)I x + (-2 ±3) =0

x = $\frac{-(-2\pm3)I\pm\sqrt{[(-2\pm3)I]^2+4(2\pm3I)}}{-2}$

x = $\frac{-1I\pm\sqrt{7\pm3I}}{-2}$ or x = $\frac{5I\pm\sqrt{13\pm3I}}{-2}$

$W^2 = \frac{[4*x^2]*[(x-1)^2]+[4*(x+1)^2]*[(x-2)^2]}{2[4(x-1)^2]*[4(x-2)^2]}$

= 0

solve($[4 * x^2] * [(x-1)^2] + [4 * (x+1)^2] * [(x-2)^2]$, x, explicit, allsolutions)

$$x = \frac{1}{2} + \frac{\sqrt{3} * \sqrt{\left(\frac{\left((251+6*\sqrt{1689})^{\left(\frac{2}{3}\right)}+5*(251+6*\sqrt{1689})^{\left(\frac{1}{3}\right)}+13\right)}{1}\right)} \Bigg/ (251+6*\sqrt{1689})^{\left(\frac{1}{3}\right)}}{6} \pm \frac{1}{6}\left[\sqrt{3} * K\right] + k_1 \text{ or}$$

$$x = \frac{1}{2} + \frac{\sqrt{3} * \sqrt{\left(\frac{\left((251+6*\sqrt{1689})^{\left(\frac{2}{3}\right)}+5*(251+6*\sqrt{1689})^{\left(\frac{1}{3}\right)}+13\right)}{1}\right)} \Bigg/ (251+6*\sqrt{1689})^{\left(\frac{1}{3}\right)}}{6} \pm \frac{1}{6}\left[I\sqrt{3} * K\right] + k_1$$

Hence, some of the w's values obtained may be real number as well as some of the w's values are complex number. But there may be a simple and interesting result as follow:

$\frac{4x^2+4(x-2)^2\frac{(x+1)^2}{(x-1)^2}}{2(4)\left(4\frac{(x-2)^2}{(x-1)^2}\right)}$, when x ⟶ ∞, both $\frac{(x+1)^2}{(x-1)^2}$ and $\frac{(x-2)^2}{(x-1)^2}$ ⟶ 1,

$\frac{4x^2+4(x-2)^2\frac{(x+1)^2}{(x-1)^2}}{2(4)\left(4\frac{(x-2)^2}{(x-1)^2}\right)} \longrightarrow \frac{4x^2+4(x-2)^2}{32}$

then ,

when x = 0 or x = 2, $\frac{4x^2+4(x-2)^2}{32} \longrightarrow \pm\frac{1}{2}$

If x = $\frac{1}{2}$, then $Prime_{j+1} - Prime_j = \frac{\left(\frac{1}{s}\right)\left(\frac{1}{j}\right)\left((2)\left(\frac{1}{\sqrt{2}}\right)(P_{j-1})+\left(\frac{1}{j}\right)\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)P_j^2P_{j-1}^2-\left[\frac{1}{2}P_j^2+\frac{1}{2}P_{j-1}^2\right]+1}$.

By taking limit j tends to infinity and $(1 / 1 + ln(j)) \leq P_j \leqslant (1 / ln(j))$, $[W^4(P_j)^2(P_{j-1})^2 - W^2[(P_j)^2 + (P_j1)^2] + 1]$ tends to 1 and $2W^* P_{j-1}$ tends to zero.

The final result of the prime gap will still be a complex number as the prime gap equal to: Prime j – Prime j-1 = {[-1 / (u+vI)*j]* (1/j)]} = $\frac{-1}{(u+vI)}$ which is still a complex number or a contradiction. There is a similar outcome for x = $-\frac{1}{2}$, this author will NOT repeat.

3.**A Revisited (or a modificaion/correction) Extended Proof to the Riemann Hypothesis by the Truth table [2], [4] & [5]:**

Beside the truth table in figure 1 [5] & [6], we also define the following truthness and falseness:

Non-trivial Zeta Zeros	True (T)	Lie on the critical line	True (T)
Other Normal Complex Number	False (F)	Lie outside the critical line	False (F)

Figure 1: An additional truth table for the proof of Riemann Hypothesis.

Case I: the (assumption) truth for the Riemann Hypothesis statement gives a positive true result and hence implying RH is correct (i.e. true & true imply true – row one of Figure 1 [5] & [6]). Or to be precise:

($\forall$ non-trivial Zeta Zeros) is true → (lie on the critical line) is true, then the prescribed (or the Riemann Hypothesis) statement is true;

In other words, we want to show for all non-trivial Riemann Zeta zeros, they must not lie outside the critical the critical line [8] & [9]**.** The proof for the Riemann Hypothesis is said to be true has been shown as in my previous paper [4] by employing Matlab programming code for the verification [3] all over the complex infinity plane except the line x = 1 which is a singularity and hence has an infinite many solutions or infinite many cases [10] & [11];

Case II: the (assumption) false for the Riemann Hypothesis statement gives a positive true result and hence implying RH is correct (i.e. false & true imply true – row two of Figure 1 [5] & [6]). Or to be precise:

($\forall$ other complex-valued numbers excluding the non-trivial zeta zeros) (i.e. negation to the non-trivial zeta zeros) is false ⟶ (lie on the critical strip) is true, then the prescribed (or the Riemann Hypothesis) statement (i.e. all non-trivial zeta zeros must lie on the critical line) is true.

In other words, we want to show that ∀ other normal complex values (i.e. by excluding those non-trivial zeta zeros), they must lie on the critical line [8] & [9]. As shown in this author's previous paper named "The Quantized Constants with Remmen's Scattering Amplitude to Explain Riemann Zeta Zeros" [1] and the former section in the present paper, the Riemann non-trivial zeta root equation is

0.5 + ($y_1 \leqslant y \leqslant y_2$) I where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$. However, the root equation is NOT the whole

x = 0.5 line or the equation 0.5 ± yI where $y is true \forall \mathbb{R}$. Actually, the imaginary part of the non-trivial zeta zeros or their y's equation is spreading between $\frac{\pm cot(k)}{ln(k)}$ and $\frac{\pm tan(k)}{ln(k)}$ but NOT for all y = $\mathbb{R}$ (that covers all of the imaginary axis when x = 0.5). Thus, we may obviously conclude that

$\frac{\pm cot(k)}{ln(k)} \leqslant y \leqslant \frac{\pm tan(k)}{ln(k)}$ and the imaginary parts of the other normal complex number -- $y \notin \left(\frac{\pm cot(k)}{ln(k)}, \frac{\pm tan(k)}{ln(k)}\right)$ constitute the whole critical strip x =0.5. Or ∀ other normal complex numbers, such that

$\{z \mid z \in 0.5 \pm \mathbb{R}/(y_1 \leqslant y \leqslant y_2)\, I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ and they lie on the x = 0.5, (with excluding those non-trivial zeta zeros). Or

$\{z = 0.5+yI \mid \Im(z) \notin (y_1 \leqslant y \leqslant y_2)I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ must lie on the critical line x = 0.5 but NOT those non-trivial zeta zeros $y \in (y_1, y_2)$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ [10] & [11].

Case III: the (assumption) true for the Riemann Hypothesis statement gives a negative false result and hence implies RH is incorrect (i.e. false & true imply false – row two of Figure 1 [5] & [6]). Or to be precise: (∀non-trivial zeros) is true → (not lie on the critical line) which is a false outcome, then prescribed (Riemann Hypothesis) statement is false.

In other words, we want to prove that for all non-trivial zeta zeros must not lie on the critical line is false. (i.e. This may imply all non-trivial zeta zeros must lie on the critical line.) [8], [9]

First, let us consider the normal complex numbers [7], [15], [16]

$\{z \mid z \in x + yI\, for\, x \neq 0.5\}$ (i.e. There are infinite many complex numbers that lie outside the critical line. Or NOT all normal complex numbers must lie on the critical line x = 0.5. In other words, there are some other kind of complex number lie on the critical line x = 0.5. But there are only two types of complex numbers, that says, normal complex numbers and non-trivial Riemann Zeta zeros. Hence, there are some non-trivial Riemann Zeta zeros existing on the critical line x = 0.5 or the existance of the non-trivial Riemann Zeta zeros lie on the critical line x = 0.5.) [10] & [11]

Although practically, we cannot directly determine whether there may be any non-trivial zeta zeros lie outside the critical line, this author notes that there are actually infinite many counter examples (or the disproof) for the above statement "**for all of the non-trivial zeta zeros must not lie on the critical line**". This is because as shown in my paper named , "The Quantized Constants with Remmen's Scattering Amplitude to Explain Riemann Zeta Zeros" [1] and the former section in the present paper, the Riemann non-trivial zeta root model equation is:

$\{z \mid z \in 0.5 \pm (y_1 \leqslant y \leqslant y_2)I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$

which are infinitely many lie on the critical line x = 0.5.

Indeed, the above Riemann Zeta non-trivial root model equations are just the (infinitely many) counter examples (or the disproof) to the aforementioned statement or "**for all of the non-trivial zeta zeros must not lie on the critical line**" [7], [15] & [16]. But we cannot find any example of the nontrivial zeta zeros lies outside the critical line. Hence, the statement for all (non-trivial zeros) is true → (not lie on the critical line), which is a false/negative (the negotation of being lie on the critical line) outcome (but NOT the statement is false), then the prescribed

(Riemann Hypothesis) statement (i.e. "**for all of the non-trivial zeta zeros must not lie on the critical line**") is said to be a false as only one counter example for disproof of the statement will be enough but now we have infinite many counter examples to disprove it.

Case IV: the (assumption) false for the Riemann Hypothesis statement gives a (negative) false result and hence implying RH is true (i.e. false & false imply true – row four of Figure 1 [5] & [6]). Or to be precise:

(for all normal complex numbers) is false (or the negation of the non-trivial zeta zeros) → (not lie on the critical line) or which is a false outcome, then prescribed (Riemann Hypothesis) statement is true. **In other words, we want to show that for all of the normal complex values, that lie outside the critical line implies the Riemann Hypothesis is true (i.e. all non-trivial zeta zeros must lie on the critical line).** [8] & [9]

The proof from the aforementioned Case III tells us that, there are NO non-trivial zeta zeros lies outside the critical line or all of the non-trivial zeta zeros must lie on the critical line. That say, there are some normal complex values $\{z \mid z \in x + yI for x \neq 0.5\}$ other than the non-trivial zeta zeros, lie outside the critical line, is correct. In fact, both of the above Case II & Case III tell us that there are both non-trivial zeta zeros and normal complex numbers $\{z \mid z \in 0.5 \pm \mathbb{R}/(y_1 \leqslant y \leqslant y_2)\ I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ lie on the critical line. However, from the Case III, all of the non-trivial zeta zeros must NOT lie outside the critical line. In other words, all of the complex numbers that lie outside the critical line must be a normal one $\{z \mid z \in x + yI for x \neq 0.5\ \}$ but NOT those that lie between the non-trivial zeta zeros $\{z \mid z \in 0.5 \pm \mathbb{R}/(y_1 \leqslant y \leqslant y_2)\ I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$. Hence, **all of the normal complex values, that lie outside the critical line is true implies, the Riemann Hypothesis, all of the nontrivial zeta zeros must lie on the critical line x = 0.5 is true.** [10] & [11] Actually, the normal complex numbers are distributed all over the real-complex plane no matter x = 0.5 or NOT 0.5 and $\{z \mid z \in 0.5 \pm \mathbb{R}/(y_1 \leqslant \Im(z) \leqslant y_2)\ I\}$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$(i.e. other normal complex numbers but NOT Riemann Zeta non-trivial zeros lie on the critical line)

Or $\{z \mid z \in x + yI for x \neq 0.5\ \}$ and $\{z \mid z = 0.5 + yI \notin (y_1 \leqslant y \leqslant y_2)I\}$where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ are actually independent of the critical strip and distribute all over the real-complex plane. On the contrary, all of the non-trivial zeta zeros or the Riemann Zeta Root model equation, 0.5 ± $(y_1 \leqslant \Im(z) \leqslant y_2)I$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ depends or lies on the critical strip x = 0.5.

(N.B. From my previous paper named "A Full & Detailed Proof" [4] case 1, it shows that if s = 1, then when we substitute it into some of the well-known Riemann equations such as $\prod_{i=1}^{\infty}(z - z_i)$= $\xi(1) = \sum_{n=1}^{\infty} 1/n = \prod_{j=1}^{\infty}\left(1 - 1/prime_j\right)^{-1}$, it will give out a fractional number which is a contradiction to the fact that the prime gap must be an integer. Similarly, for the same paper's case 2, with ξ(s) where {s = u + v*I and u, v are real numbers with I = $\sqrt{-1}$}/{0.5+y*I) for some y belongs to real}, the prime gap will be a complex number which is also a contradiction to the fact that it must be an integer. Then there must be no non-trivial zeta zeros lie outside the critical line

x = 0.5. In addition, for s = 1, it is actually a singularity for the Riemann Zeta function ξ(s). Finally, as described in my present paper's previous sections, for another critical line x = 1.5, it should be rejected. This is because all of the contour integrals computed from the Matlab code by the "A Vertification" [3] give us those residuals are just zeros without any multiples of the π. Thus, we may have the following position map in the figure 2 [27], for the most recent known important locations as depicted aforementioned.)

To sum up, this author may have used the skill of mathematical-linguistics, Riemann Zeta Root Model Equation -- $\frac{1}{2} \pm \Im(z) I$ where Im(z) spreads between $\frac{\pm cot(k)}{ln(k)}$ and $\frac{\pm tan(x)}{ln(x)}$ for $0 \le k \leqslant \infty$ together with the assistance of the truth table

(figure 1 [5], [6] & also figure 1 of this paper), to prove the correctness of the Riemann Hypothesis. As shown from the aforementioned Case I to Case IV together with the first two case in the [5] , this author thus have confidence to propose that the Riemann Hypothesis is true or all of the Riemann non-trivial zeta zeros must lie on the critical strip x = 0.5.

Hence, we may conclude and have the following theorem:

"All of the non-trivial zeta zeros must lie on the critical strip x =0.5 if and only if there are infinite many counter examples $0.5 \pm (y_1 \leqslant \Im(z) \leqslant y_2) I$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ and $0 < k < 2\pi$,

for the disproof to the non-trivial zeta zeros lies outside the critical strip x = 0.5".

In practice, we prove the above theorem by [1] – [5] and the present paper:

" If part":

All non-rivial zeta zeros lie on the critical implies non-existence of non-trivial zeros lie outside the x = 0.5. Or there are inifinite many counter examples for the disproof of the non-trivial zeta zeros lie

outside the critical strip or $0.5 \pm (y_1 \leqslant \Im(z) \leqslant y_2) I$ where $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ and $0 < k < 2\pi$.

" Only-if part":

Already proved in the present paper's Case III.

Therefore, obviously, the Riemann Hypothesis has been proved to be true.

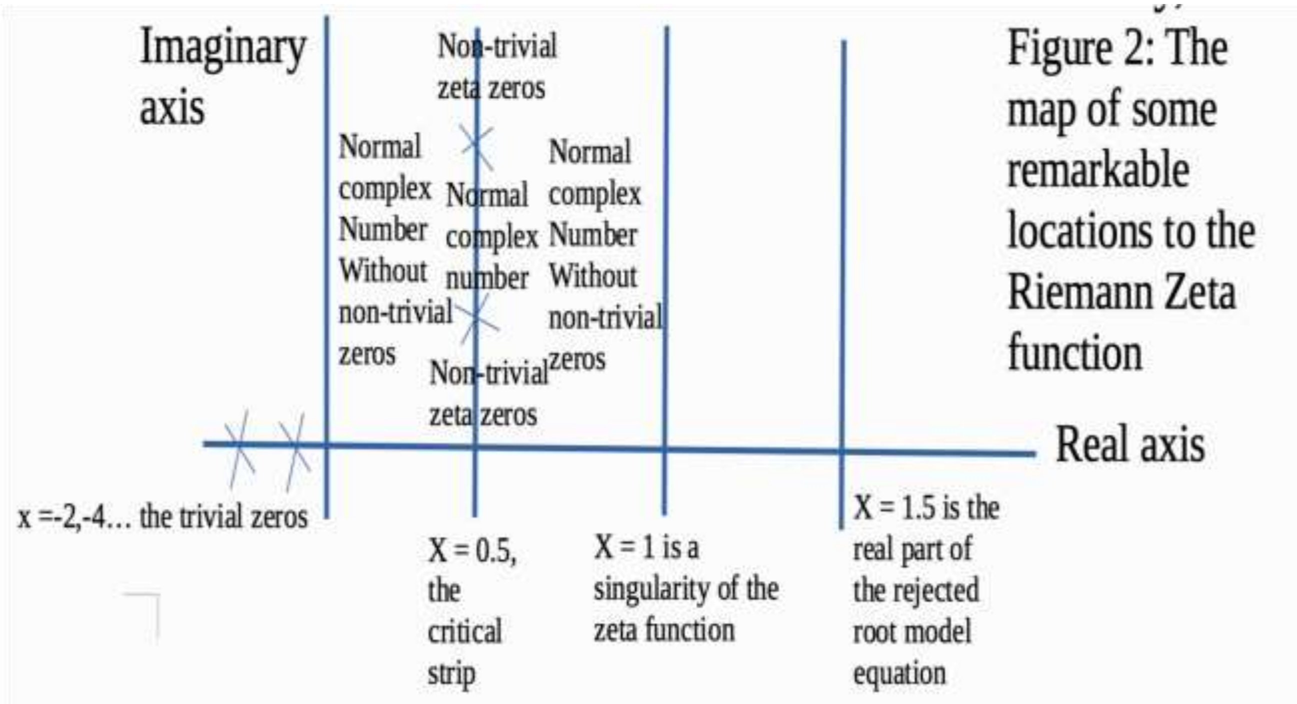

Figure 2: The map of some remarkable locations to the Riemann Zeta function

Hypothesis X	Consequent Y	Conditional X → Y	Reason(s) for the true/false of the conditional & the implication(s) to RH
True (T)	True (T)	True (T)	Matlab-Code Verification [3]
False (F)	True (T)	True (T)	The best Riemann Zeta non-trivial Root model equation is: $\{z = x + yI \mid z \in 0.5 \pm y_1 \leqslant y \leqslant y_2 I\}$, there are other complex numbers such that $\{z = 0.5+yI \mid \Im(z) \notin y_1 \leqslant y \leqslant y_2 I\}$ lie on the critical strip x = 0.5.

True (T)	False (F)	False (F)	There are infinite many counter examples {z | $z \in 0.5 \pm y_1 \leqslant y \leqslant y_2 I$ }to the fact that non-trivial zeta zeros must lie outside the critical strip x= 0.5. Otherwise, any outside critical strip non-trivial zeta zeros -- ξ(1) & ξ(s) where s = {z | z = u + vI & $z \notin 0.5 \pm yI$ for u, $v \in \mathbb{R}$ } will lead to either a fractional or a complex prime gap value contradictions. (Case I & Case II in [4]. In fact, we still CANNOT find any example(s) /value(s) /evidence(s) for the non-trivial zeta zeros to be lie outside x = 0.5 even we employ the Matlab code in [3] for any essential & necessary verification.) We may conclude there is no non-trivial zeros lie outside x = 0.5.
False (F)	False (F)	True (T)	Only {z | $z \in x + yI for x \neq 0.5$ } lies outside the critical strip x = 0.5 without any non-trivial zeta zeros but NOT {z | $z \in 0.5 \pm \mathbb{R}/(y_1 \leqslant y \leqslant y_2) I$}lies on the critical strip x = 0.5.

Table 3: The 4 cases statement truth table and their summerized reasons for why & how to determine the "Truthness of the Riemann Hypothesis" for $y_1 = \frac{\pm cot(k)}{ln(k)}$, $y_2 = \frac{\pm tan(k)}{ln(k)}$ & 0<k<2π.

(N.B. In fact, logarithm is the mirror image inverse of the poly-logarithm and exponential is also the mirror image inverse of the logarithm. This result implies that we may express the Riemann Zeta function (a type of poly-logarithm) in terms of an exponential (Taylor) series. We may apply the Taylor approximation series method for the Riemann Zeta function $\sum_{n=1}^{\infty}\frac{1}{n^s}$ as it is differentiable for s < 1 but NOT equal to 1 although it is well known for the sum of the Zeta function's divergence over s < 1 and s equal to 1. In reality, the Riemann Zeta function has a singularity at s = 1 or $\sum_{n=1}^{\infty}\frac{1}{n}$ which is undefined (i.e. divergent or tends to an infinity). In practice, when we treat the value of the improper integral $\sum_{n=1}^{\infty}\frac{1}{n}$ as a black-box, then we may avoid the undefined value for the computation to the above equation for the division in 1 / $\sum_{n=1}^{\infty}\frac{1}{n}$ or $\frac{1}{\infty}$ may be viewed as a kind of "dark art". In addition, when n = 0, the Riemann Zeta function $\sum_{,}\frac{1}{n^s}$ will also tend to an infinity or undefined. In such of both cases, we cannot apply the Taylor Approximation series at n = 0. Thus, my previous determined nontrivial zeta root model equation 0.5 +/- 4*cot(ln(x))/(x+1)^2 becomes invalid. In reality, the harmonic series $\sum_{n=1}^{N}\frac{1}{n}$= log N may be trapped between y = $\sum_{n=0}^{N}\frac{1}{n+1}$ and y = 1 + $\sum_{n=1}^{N}\frac{1}{n}$[14] while both $\sum_{n=0}^{N}\frac{1}{n+1}$ and 1 + $\sum_{n=1}^{N}\frac{1}{n}$ are smooth functions and thus can be differentiated infinitely. Hence, one may apply the corresponding commercial software Maple computed Taylor series to the harmonic series $\sum_{n=1}^{N}\frac{1}{n}$ successfully as the logarithmic function is indefinitely differentiable.)

A Conclusion for the Proof to the Riemann Hypothesis

In a nutshell, this author may have solved the Riemann Hypothesis problem. In fact, this author handles the problem by the following algorithm or procedure:

1. Employ the commercial mathematical software such as the Canada's Maple-soft to find the root model for the Riemann Zeta Function;
2. Use the telescopic logarithmic method to determine if there may be any contradictions for the prime gap difference;
3. Develop a Mat-lab coding for checking the contour integral in order to approximately locate the non-trivial roots of Riemann Zeta function;
4. Apply the mathematical-linguistic (truth table) method to prove and disprove the statement of Riemann Hypothesis and hence conclude the truth of the Riemann Hypothesis;
5. Conclusive summary of how & why the Riemann Hypothesis has been solved.

Practically, proof or disproof the Riemann Hypothesis statement is NOT an easy task. One may need to overcome lots of barriers in the aforementioned procedure. In fact, rather than proving the RH statement, this author also finds that there may be some structures lying between the x = 0.5, x = 1 and x = 1.5. In addition, there are also some other structures or patterns for the randomness of those non-trivial zeta zeros together with the dual of these zeros or the prime numbers. This author hopes through the research of the Riemann Hypothesis conjecture and if conditions are available, this author or those interested parties may further investigate the structures to the random-ness of the non-trivial zeta zeros together with their prime duals, then, we may finally develop a novel system of the quantum lattice cryptography or at least have some contributions for such a proposed issue.

In reality, this author concludes that he may have solved the Riemann Hypothesis conjecture by the above prescribed five steps. In fact, the practical Riemann non-trivial Zeta root model equation is: {z = x + yI | $z \in 0.5 \pm \left(\frac{\pm cot(x)}{ln(x)} \leqslant y \leqslant \frac{\pm tan(x)}{ln(x)}\right) I$ where 0<x<2π}. At the same time, $0.5 \pm \frac{4cot(ln(x))}{(x+1)^2}$ is another best approximation or the artifical one to the Riemann zeta function's root model equation. We shall discuss the issue of how to the optimize the artifical model equation together with the elliptical problem of the (quantum) lattice for the topic of cryptography in my next paper of the series. Certainly, with reference to my paper in algorithmic continuum flow chart, we may develop the corresponding one for the Riemann Hypothesis according to the aforementioned 4 amended cases plus the 2 previous cases [5]. Then we may develop the respective computer programming code to simulate the RH problem solver's six cases. However, such issuse is out of the focus of the present paper, this author will leave it to those interesting parties.

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