ISSN: 3065-0577

International Journal of Data science and Statistics

Research Article

A NOVEL INTEGER SOLUTION SET FOR THE HOMOGENEOUS CUBIC EQUATION $X^3 + Y^3 = 42ZW^2$

Dr. R. Karthikeyan

Associate Professor, Department of Mathematics,School of Engineering and Technology,Dhanalakshmi Srinivasan University, India.

Abstract

This paper aims at presenting different choices of non-zero solutions in integers to the homogeneous cubic equation with four unknowns given by $x^3 + y^3 = 42zw^2$. Various sets of integer solutions are obtained by Scrutiny, substitution technique and method of factorization.

Keywords: Homogeneous cubic equation, Cubic equation with four unknowns and Integer solutions

Introduction

Diophantine equations, one of the areas of number theory, occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. It is well-known that Diophantine equations are rich in variety. Particularly, finding integer solutions to homogeneous cubic equation with four unknowns is a topic of current research. While collecting problems on the same, the article presented in [1] was noticed and the authors have obtained a few sets of integer solutions .However, there are many more fascinating patterns of solutions in integers. The main thrust of this paper is to exhibit other solution patterns to the

Homogeneous cubic equation with four unknowns $x^3 + y^3 = 42zw^2$.

Technical procedure Consider the homogeneous cubic equation with four unknowns given by

$X3 + y^3 = 42zw^2$ (1)

At the outset, by scrutiny, it is seen that (1) is satisfied by the following integer quadruples (x,y,z,w) = $(10,8,9,2),(-1,-5,-3,1),(8,-2,3,2),(16,-4,6,4),\ {}^{s}m(m^3+n^3),42^{s}n(m^3+n^3),42^{s-1}(m^3+n^3)^2,42^{s}(m^3+n^3))$ (42

However, there are some more choices of integer solutions to (1) that are illustrated below:

ISSN: 3065-0577
International Journal of Data science and Statistics
Research Article

The substitution of the linear transformations

$x = u + v, y = u - v, z = u, u \square v \square 0$ (2)

In (1) leads to the ternary quadratic equation

$u^2 + 3v^2 = {}^1w^2$ (3)

Illustration 1 Let

$w = 4(a^2 + 3b^2)$ (4)

The integer 21 on the R.H.S. of (3) is written as

$= 18(a^2 - 3b^2) - 12ab,$

$u = 2(a^2 - 3b^2) + 36ab.$

In view of (2), one has

$x = 20(a^2 - 3b^2) + 24a\,b,$

$y = 16(a^2 - 3b^2) - 48\,ab,$ (6)

$z = 18(a^2 - 3b^2) - 12\,ab.$

Thus, (4) and (6) satisfy (1).

Illustration 2 Write (3) as

$u^2 + 3v^2 = 21w^2 * 1$(7)

Let the integer 21 on the R.H.S. of (7) be written as

$$1 = \frac{(9 + i\sqrt{3})\,(9 - i\sqrt{3})}{4} \quad (5)$$

Using (4) & (5) in (3) and employing the method of factorization ,consider

$u + i\,3v = 2\,(9 + i\,3)\,(a + i\,3b)^2$

On equating the coefficients of corresponding terms ,we have

$$21 = (3+i2\sqrt{3})(3-i2\sqrt{3}) \quad (8)$$

Consider the integer 1 0n the R.H.S. of (7) as

$$1 = \frac{(1+i4\sqrt{3})\,(1+i4\sqrt{3})}{49} \quad (9)$$

Assume

$$w = (a^2 + 3b^2) \quad (10)$$

Using (8), (9) & (10) in (7) and employing the method of factorization, consider

$$u + i\sqrt{3}v = \frac{(1+i4\sqrt{3})}{7}(3+i2\sqrt{3})(a + i\sqrt{3}b)^2$$

On equating the coefficients of corresponding terms, we have

$u = -3(a^2 - 3b^2) - 12ab,$

$v = 2(a^2 - 3b^2) - 6ab.$

In view of (2), one has

$x = -(a^2 - 3b^2) - 18ab,$

$y = -5(a^2 - 3b^2) - 6ab, \quad (11)$

$z = -3(a^2 - 3b^2) - 12ab.$

Thus, (10) and (11) satisfy (1).

Note 1

It is to be seen that, apart from (9), we have

ISSN: 3065-0577
International Journal of Data science and Statistics
Research Article

$$1=\frac{(3r^2-s^2+i2rs\sqrt{3})\,(3r^2-s^2-i2rs\sqrt{3})}{(3r^2+s^2)^2}$$

Repeating the above process, a different set of integer solutions to (1) is obtained.

Illustration 3

Rewrite (3) as

$$21w^2-u^2=3\,v^2 \quad (12)$$

Assume

$$v=21a^2-b^2 \quad (13)$$

Express the integer 3 on the R.H.S. of (12) as

$$3=\frac{(\sqrt{21}+3)\,(\sqrt{21}-3)}{4} \quad (14)$$

Using (13) & (14) in (12) and employing the method of factorization, consider

$$\sqrt{21}w+u=\frac{(\sqrt{21}+3)}{2}(\sqrt{21}a+b)^2$$

On equating the coefficients of corresponding terms, we have

$$u=\frac{3(21a^2+b^2)+42ab}{2},$$

$$w=\frac{(21a^2+b^2)+6ab}{2}$$

As our aim is to obtain integer solutions, replacing a by 2A and b by 2B, We have

$u = 6(21A^2 + B^2) + 84AB,$

$v = 4(21A^2 - B^2),$

And

$w = 2(21A^2 + B^2) + 12AB$ (15) In view of (2) , we have

$x = 210A^2 + 2B^2 + 84AB,$

$y = 42A^2 + 10B^2 + 84AB,$ (16)

$z = 6(21A^2 + B^2) + 84AB.$

Thus, (15) and (16) satisfy (1).

Note 2

It is to be seen that, apart from (14), we have

$3 = (2\sqrt{21}+9)(2\sqrt{21}-9)$

Repeating the above process, a different set of integer solutions to (1) is obtained.

Illustration 4

Rewrite (3) as

$21w^2 - 3v^2 = u^2 * 1$ (17)

Assume

$u = 21a^2 - 3b^2$ (18)

Express the integer 1 on the R.H.S. of (17) as

Consider

$$1 = \frac{(\sqrt{21} + 2\sqrt{3})(\sqrt{21} - 2\sqrt{3})}{9} \quad (19)$$

Using (18) & (19) in (17) and employing the method of factorization ,

ISSN: 3065-0577
International Journal of Data science and Statistics
Research Article

$$\sqrt{\quad} \; 21w\sqrt{\quad} + \frac{(\sqrt{21}+2\sqrt{3})}{3} (\sqrt{21}A+\sqrt{\quad} \; 3v = 3b)^2$$

On equating the coefficients of corresponding terms, we have

$v = 14a^2 + 2b^2 + 14ab$,

And

$w = 7a^2 + b^2 + 4ab$.

In view of (2), we have

$x = 35a^2 - b^2 + 14ab,\ y = 7a^2 - 5b^2 - 14ab$,

$z = .21a^2 - 3b^2$

Thus, the above values of x,y,z,w satisfy (1).

Note 3

It is to be seen that, apart from (19), we have

$$1 = \frac{(2\sqrt{21}+5\sqrt{3})(2\sqrt{21}-5\sqrt{3})}{9}$$

Repeating the above process, a different set of integer solutions to (1) is obtained.

Illustration 5

Express (3) in the form of ratio as

u +3w = 3(2w − v) = P , Q □ 0 (20)

——— ————— —

2w + vu −3w Q

Solving the above system of double equations, one has

$u = 3P^2 + 12PQ - 9Q^2$,

(21)

$v = -2P^2 + 6PQ + 6Q^2$,

And

$w = 3Q^2 + P^2$ (22)

From (21) and (2), we have

$x = P^2 + 18PQ - 3Q^2$,

$y = 5P^2 + 6PQ - 15Q^2$, (23)

$z = 3P^2 + 12PQ - 9Q^2$.

Thus, (1) is satisfied by (22) and (23).

Conclusion

In this paper, we have illustrated various ways of obtaining non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns given by $x^3 + y^3 = 42zw^2$ and these solutions are different from the solutions presented in [1]. To conclude, one may attempt for getting integer solutions to other choices of homogeneous cubic equations with four unknowns.

References

S.A.Shanmugavadivu and R.Anbuselvi, On the Homogeneous Third Degree Diophantine Equation with Four Unknowns $x^3 + y^3 = 42zw^2$, Turkish Journal of Computer and Mathematics Education, Vol.12, No.11 (2021), 4559-4564.